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INFLUENCE OF AN ADDITIVE CONSTANT ON LIPSCHITZ CONDITIONS: A METRIC PROJECTION APPROACH

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Abstract- To rigorously analyse the extremal bounds of the Lipschitz constant in metric projections over totally convex subsets of Banach spaces and assess its implications in variation analysis and control. A variation framework based on non-expansive mappings and projection operators is developed. The analysis employs advanced tools from convex analysis and functional geometry, introducing a supplementary parameter satisfying a modified Lipschitz condition. Comparative analysis is performed between standard and modified projection schemes across varying convex subsets in infinite-dimensional Banach spaces. The study establishes novel theoretical bounds—both upper and lower—for the Lipschitz constant governing metric projections in totally convex subsets of Banach spaces. The introduction of an

additional regularizing constant enhances the local stability and smoothness properties of Lipschitz continuous functions, especially under

strong convexity conditions. Results reveal that the modified projection scheme not only preserves non-expansiveness but also improves convergence behaviour in iterative methods. Compared to classical results, this framework provides tighter control over deviation in optimization and variation inequality problems. The analytical results are consistent with, yet significantly extend, foundational work in nonlinear functional analysis, offering deeper insight into the geometry-driven behaviour of metric projections in high-dimensional analysis.

Derives tighter Lipschitz bounds for projections via a modified constant, enhancing smoothness and convergence in nonlinear optimization over Banach spaces.

Keywords: Lipschitz Condition, Metric Analysis, Strong Projection, Banach Space, Convex Sets, Optimization

I. INTRODUCTION

The Lipschitz condition is a term used in mathematics to characterize the rate of change of a function, especially when examining the behavior of functions. A constant factor restricts how quickly the function's output can alter in response to changes in its input for a function that meets the Lipschitz condition. A function that converts input values to output values might be used as an example. The pace at which the output changes as the input changes is constrained by a fixed constant if this function is in accordance with the Lipschitz criterion. In other words, the function must vary in a controlled way over its domain and does not show abrupt jumps or unbounded expansion.

A Lipschitz continuous function can be comprehended by looking at its graph. The function's graph will be devoid of steep slopes and abrupt corners if it meets the Lipschitz criterion. Rather, a constant (the Lipschitz constant) regulates the rate of change at each point in the graph, causing it to change smoothly and gradually. A useful tool for evaluating the stability and smoothness of functions is the Lipschitz condition. It is essential to several branches of mathematics, such as optimization, machine learning, and analysis. A key idea in differential equations and mathematical analysis, the Lipschitz condition is especially important when discussing the stability and uniqueness of solutions to specific issues. When there is a constant K that is such that, for every point x and y in the domain, the following statement is true: the difference in the values of their functions is bounded by K times the difference between x and y. This is the case when there is a constant K. In the event that this is the case, then the function is said to be in compliance with the Lipschitz criterion.

Influence of additive Constant on the Lipschitz Condition

For this study, it is presumed that the section B is a genuine Banach space, and the notation B* is used to indicate the dual space that is included within this section. According to this configuration, A real Hilbert space is denoted as H, while the n-dimensional Euclidean space is referred to as Rn. Both of these spaces are n-dimensional. The statement implies that the evaluation of the functional P ∈ B at the position u ∈ B is currently being carried out. This is the consequence of the statement. For the purpose of providing a description of the ball Br(l), one may say... In order to acquire the set Br(l), it is necessary to define the set ϰu∈ B ||∑u− l ||≤r}. When set A is a subset of set B, the symbols $\partial\mathcal{A}$, int $\mathcal{A}$, and cl A are utilized to represent construct A's perimeter, interior, and boundary, in that sequence. This is because these symbols are used to denote the subset of set B. The diameter of a subset $\boldsymbol{z} \subset$ B can be determined using the following formula: Diam $\mathcal{A}$ = sup_{u, y ∈ A} ‖u − y‖_∞. The following illustrates the component of the support function tasked with delineating the subset A⊂ B:

$$s(\mathcal{P},\mathcal{A}) = \sup_{\ell\in\mathcal{A}}(\mathcal{P},\ell), sup(p,\ell), \forall p \in \mathfrak{B}*.$$

The typical cone that leads to a convex closed subset A⊂B at a point ℓ ∈A is represented by the notation N(A, ℓ). Its definition is as follows:

$$N(\mathcal{A},\ell) = \{\mathcal{P} \in \mathfrak{B}^* \mid (\mathcal{P},\ell) \geq s(\mathcal{P},\mathcal{A})\}.$$

For any closed convex subset $\mathcal{A}$ that belongs to $\mathfrak{B}$ and for every vector P that is an element of $\mathfrak{B}$*, the set $\mathcal{A}$(P) is defined as the collection {x∈$\mathcal{A}$|(P,u)=s(P, $\mathcal{A}$)}. where s(P,$\mathcal{A}$) indicates a particular value that is related with P and $\mathcal{A}$.

In the following formula, the distance between a point u that belongs to the set $\mathfrak{B}$ and the subset $\mathcal{A}$ that is subset of x is specified.

$$\mathfrak{C}(u,\mathcal{A}) = \inf_{\ell\in\mathcal{A}} \| u,\ell \|$$

A point u∈$\mathfrak{B}$ is defined as the metric projection onto the set $\mathcal{A}$.

$$\mathcal{P}_{\mathcal{A}}u = \{\ell \in \mathcal{A} \mid \| u-\ell \| = \mathfrak{C}(x,\mathcal{A})\}.$$

For any subset $\mathcal{A}\subset\mathfrak{B}$, the open $\mathfrak{C}$-neighbourhood of $\mathcal{A}$, defined as:

$$U(\mathcal{A},\mathfrak{C}) = \{u\in\mathfrak{B} \mid \mathfrak{C}(u,\mathcal{A}) < \mathfrak{C}\}.$$

For every closed convex subset A that belongs to the set H and for every point u that belongs to the set H, the set $\mathcal{P}_{\mathcal{A}}u$ is a singleton, as is well known. Moreover, the following inequality is true for any pair of points u_0 and u_1 that are valid in H:

$$\|\mathcal{A}_0,\ \mathcal{A}_{1,}\mathcal{A}_2,\ \mathcal{A}_{3,}\mathcal{A}_4,\ \mathcal{A}_{5,}\| \leq 1\ \|u_0,\ u_{1,}u_2,\ u_{3,}u_4,\ u_{5,}\|,$$

where $\mathcal{A}_i = \mathcal{P}_{\mathcal{A}}u_i$ for i=0,1,2,3,4,5

(1.1)

In Formula (1.1), The Lipschitz criteria of 1 serves as the most optimal bound in a general context, and it can only be obtained for closed affine subspaces. This is the only way to obtain it. Due to this, the limits that can be attained are restricted. Because the condition is met, this is the case. Alternatively, let $\mathcal{A} = B_{\mathcal{R}}(\ell) \subset H$, and consider any points $u_0, u_{1,} u_2, u_{3,} u_4,\ u_{5,} \in H$ with $\rho(u_0, \mathcal{A}) = \rho_0 > 0$ and $\rho(u_1, \mathcal{A}) = \rho_1 > 0$, $\rho(u_2, \mathcal{A}) = \rho_2 > 0$, $\rho(u_3, \mathcal{A}) = \rho_3 > 0$, $\rho(u_4, \mathcal{A}) = \rho_4 > 0$, $\rho(u_5, \mathcal{A}) = \rho_5 > 0$ Using the cosine rule for the triangle $u_0 u_1 \ell$

(where the side lengths are $\| \ell - u_0 \| = \mathcal{R} + \rho_0$and

$\| \ell - u_1 \| = \mathcal{R} + \rho_1$, $\| \ell - u_2 \| = \mathcal{R} + \rho_2$, $\| \ell - u_3 \| = \mathcal{R} + \rho_3$, $\| \ell - u_4 \| = \mathcal{R} + \rho_4$, $\| \ell - u_5 \| = \mathcal{R} + \rho_5$) we can readily derive that:

$$\| \ell_0 - \ell_5 \| = \frac{\mathcal{R}}{\sqrt{(\mathcal{R} + \mathfrak{C}0)(\mathcal{R} + \mathfrak{C}1)(\mathcal{R} + \mathfrak{C}2)(\mathcal{R} + \mathfrak{C}3)(\mathcal{R} + \mathfrak{C}4)(\mathcal{R} + \mathfrak{C}5)}} \cdot \sqrt{\| u_0 - u_5 \|^2 - \mathfrak{C}0 - \mathfrak{C}1 - \mathfrak{C}2 - \mathfrak{C}3 - \mathfrak{C}4 - \mathfrak{C}5)^2}$$

(1.2)

Here $\ell_i = \mathcal{P}_{\mathcal{A}}, i = 0,1,2,3,4,5$

Hence, if there exists a value $\mathfrak{C} > 0$ such that $\mathfrak{C} = (u_0, B_{\mathcal{R}}(\ell) \geq \mathfrak{C}$,i = 0, 1,2,3,4,5 then

$$\| \ell_0 - \ell_5 \| \leq \frac{\mathcal{R}}{(\mathcal{R} + \mathfrak{C}0 + \mathfrak{C}1 + \mathfrak{C}2 + \mathfrak{C}3 + \mathfrak{C}4 + \mathfrak{C}5)} . \| u_0 - u_5 \| \qquad (1.3)$$

The purpose of this investigation is to determine convex closed $\mathcal{A} \subset H$ that are in accordance with the following condition: for every non-zero $\mathfrak{C}$, there is a constant $0 < (\alpha + \beta + \gamma) < C < 1$, such that for all five points $u_0, u_{1,} u_2, u_3 u_4, u_5 \in H \setminus U(\mathcal{A}, \mathfrak{C})$, the following inequality is satisfied:

$$\| \ell_0 - \ell_5 \| \leq \alpha \| u_0 - u_1 \| + \beta \| u_2 - u_3 \| + \gamma \| u_4 - u_5 \| < C \| u_0 - u_5 \|$$

$$,\ell_i = \mathcal{P}_{\mathcal{A}} u_i, i = 0,1,2,3,4,5$$

(1.4)

The discussion focuses on specific results related to the boundaries of the Lipschitz constant projection operator for this metric, building upon the discoveries reported in [1]. The analysis is guided by the concept of a highly convex set characterized by a specified radius R.

The initial approach involves examining a convex strongly subset with a radius R>0. A strongly convex subset is defined as a nonempty set that can be represented as the intersection of closed balls, where each ball has a radius R>0. Here, R denotes a strictly positive real number, which plays a critical role in defining the geometric properties of the subset. This characterization implies that the subset A is not only a subset of B but also satisfies a specific convexity condition. More explicitly, the set A is considered convex strongly if £ a subset U⊆B that meets the criteria of strong convexity and is recognized as such within the broader set B. This foundational understanding forms the basis for further exploration and analysis of the Lipschitz properties of the metric projection operator under the constraints imposed by these geometric conditions.

$$\mathcal{A} = \cap_{u\in U} B_{\mathcal{R}}(u).$$

An effective example of this is the close relationship that exists between strongly convex sets with a radius of R and symmetrically maximal sets, as well as sets with constant width. A significant relationship exists between these sets and various classical concepts. Furthermore, they constitute a critical element in various optimization and optimal control problems [17, 18, 5,6,9, 13]. For further details on strongly convex sets with a value of R greater than 0, please refer to sources [2,3,17,18,5,11,10,4,13].

Proposition 1.1 Providing Support for the Principle [2,18]: If the following condition is met for every unit vector P that is a part of H and the point { ℓ(P)} is equal to $\mathcal{A}$ (P), then a closed convex subset A that is included inside the set H is considered to be highly convex with a radius of R. This is because the point {l(P)} is equal to $\mathcal{A}$ (P).

$$\mathcal{A} \subset B_{\mathcal{R}}(\ell(\mathcal{P}) - \mathcal{R}_{\mathcal{P}})$$

Proposition 1.2 [8]: A ⊂ H be convex closed subset is defined as highly convex with radius R if and only if the following condition is met. This is the sole factor that enables us to make this assertion. For all unit vectors P and q that belong to H, and the corresponding points { ℓ (P)} = A(P) and { ℓ (q)} = A(q), a convex closed subset A is defined to be convex strongly with a range R. The term " convex strongly " refers to a specific mathematical property of a function.

$$\| \ell(\mathcal{P}) - \ell(q) \| \leq \mathcal{R} \| \mathcal{P} - q \|$$

II. ANALYSIS OF THE METRIC SYSTEM AND A ROBUST PROJECTION OF THE RADIUS R

Lemma 2.1: Consider the following: A⊂H is a convex closed subset, C is greater than zero, and C is a subset of the interval (0,1). Assume that Formula (1.4) is valid for all $u_0, u_1, u_2, u_3 u_4, u_5$∈H \ U($\mathcal{A}$,$\mathfrak{C}$), and that $\ell_i = \mathcal{P}_{\mathcal{A}} u_i, i = 0,1,2,3,4,5$.. The set A is considered to be bounded under these conditions.

Proof: It has been stated by Danford and Schwartz [14] that there is a subset which is dense S $\subset \partial$A that is such that every point $\ell \in$S satisfies the criterion.

$\ell = \mathcal{P}_{\mathcal{A}} u_\ell$. Furthermore,

$$l(\ell, u_\ell) = \{\ell + \lambda(u_\ell - \ell) \mid \lambda \geq 0\} \subset \ell + N(\mathcal{A}, \ell)$$

Therefore $\mathcal{P}_{\mathcal{A}} l(\ell, u_\ell) = \{\ell\}$. Take any points Fix ℓ, b $\in$ S. £ points $z_\ell \in$ l(ℓ, u_ℓ) and zb $\in$ l(b, u_b) sustaining

$\| z_\ell - \ell \| = \| z_b - b \| = \mathfrak{C}$ Then,

$$\| \ell - b \| \leq C \| z_\ell - z_b \| \leq C (\| z_\ell - \ell \| + \| \ell - b \| + \| z_b - b \|)$$

$$= C(2\mathfrak{C} + \| \ell - b \|),$$

Thus, $\| \ell - b \| \leq \frac{2C_{\mathfrak{C}}}{(1-C)}$. The property cl $S = \partial\mathcal{A}$ ensures that diam $\mathcal{A} \leq \frac{2C_{\mathfrak{C}}}{(1-C)}$. Consequently, by Lemma 2.1, we can restrict our attention to bounded sets.

Theorem 2.1: Consider, a convex, closed, and subset which is bounded as $\mathcal{A} \subset$ H. Let , $\mathfrak{C} > 0$ and C$\in$(0,1). If (1.4) holds for all these points $u_0, u_1, u_2, u_3, u_4, u_5 \in$H \ U($\mathcal{A}$,$\mathfrak{C}$) with $\ell_i = \mathcal{P}_{\mathcal{A}} u_i, i = 0,1,2,3,4,5$then $\mathcal{A}$ is strongly convex with a radius $\mathcal{R} \leq \frac{C_{\mathfrak{C}}}{(1-C)}$.

Proof. Let $u_0, u_1 \in \partial(H \setminus U(A, C)$, i.e. , $\mathfrak{C}(u_i, \mathcal{A}) = \mathfrak{C}, i = 0, 1.2.3.4.5$ We have

$$\| \ell_0 - \ell_5 \| \leq C \| \ell_0 - \ell_5 \| + C_{\mathfrak{C}} \| \frac{(u_0 - \ell_0)}{(\mathfrak{C})} - \frac{(u_1 - \ell_1)}{(\mathfrak{C})} - \frac{(u_2 - \ell_2)}{(\mathfrak{C})} - \frac{(u_3 - \ell_3)}{(\mathfrak{C})} - \frac{(u_4 - \ell_4)}{(\mathfrak{C})} - \frac{(u_5 - \ell_5)}{(\mathfrak{C})} \|$$

(2.5)

$$\| \ell_0 - \ell_5 \| \leq \mathcal{R} \| \frac{(u_0 - \ell_0)}{(\mathfrak{C})} - \frac{(u_1 - \ell_1)}{(\mathfrak{C})} - \frac{(u_2 - \ell_2)}{(\mathfrak{C})} - \frac{(u_3 - \ell_3)}{(\mathfrak{C})} - \frac{(u_4 - \ell_4)}{(\mathfrak{C})} - \frac{(u_5 - \ell_5)}{(\mathfrak{C})} \|$$

and, $\frac{(u_i - \ell_i)}{(\mathfrak{C})} \in N(\mathcal{A}, \ell_i)$, i=0,1,2,3,4,5

Now, we show that,

$$\left\{\frac{(u - \mathcal{P}_{\mathcal{A}} u) \mid u \in \partial (H \setminus U(\mathcal{A}, \mathfrak{C})}{(\mathfrak{C})}\right\} = \partial B1(0).$$

(2.6)

This presence $\frac{(u - \mathcal{P}_{\mathcal{A}} u) \in \partial B_1(0)}{(\mathfrak{C})}$ is clear for any point $u \in \partial$ (H \ U($\mathcal{A}$, $\mathfrak{C}$).

Consider,$\mathcal{P} \in \partial B_1(0)$ Then, there exists a point $\ell(\mathcal{P}) \in \mathcal{A}$. such that $(\mathcal{P}, \ell(\mathcal{P})) = s(\mathcal{P}, \mathcal{A})$ [17]. This implies$(\mathcal{P}, \ell(\mathcal{P})) =$

$s(\mathcal{P},\mathcal{A})$. Hence $u(\mathcal{P}) = \ell(\mathcal{P}) + \mathfrak{C}\mathcal{P} \in \partial\,(H \setminus \mathrm{U}(\mathcal{A},\mathfrak{C}))$ and $\mathcal{P}_{\mathcal{A}}(u(\mathcal{P})) = \ell(\mathcal{P})$.

Thus $\mathcal{P} = \frac{u(\mathcal{P})-\ell(\mathcal{P})}{\mathfrak{C}}$. Using Methods (2.5) and (2.6), along with Proposition 1.2, we conclude that the set $\mathcal{A}$ is strongly convex with radius $\mathcal{R}$.

Theorem 2.2. Consider convex strongly which is subset $\mathcal{A} \subset H$ of radius $\mathcal{R} > 0$. Let $u_0, u_1 \in H\setminus\mathcal{A}$, $\mathfrak{C}_i\mathfrak{C}(u_i,\mathcal{A})$, and $\ell_i = \mathcal{P}_{\mathcal{A}} u_i, i = 0, 12{,}3{,}4{,}5$. Then

$$\| \ell_0 - \ell_5 \| = \frac{\mathcal{R}}{\sqrt{(\mathcal{R}+\mathfrak{C}0)(\mathcal{R}+\mathfrak{C}1)(\mathcal{R}+\mathfrak{C}2)(\mathcal{R}+\mathfrak{C}3)(\mathcal{R}+\mathfrak{C}4)(\mathcal{R}+\mathfrak{C}5)}} \cdot \sqrt{\| u_0 - u_5 \|^2 - \mathfrak{C}0 - \mathfrak{C}1 - \mathfrak{C}2 - \mathfrak{C}3 - \mathfrak{C}4 - \mathfrak{C}5)^2)}$$

(2.7)

Proof. Using Proposition 1.2 we get,

$$\| \ell_0 - \ell_5 \| \leq \mathcal{R} \left\| \frac{(u_0-\ell_0)}{(\mathfrak{C})} - \frac{(u_1-\ell_1)}{(\mathfrak{C})} - \frac{(u_2-\ell_2)}{(\mathfrak{C})} - \frac{(u_3-\ell_3)}{(\mathfrak{C})} - \frac{(u_4-\ell_4)}{(\mathfrak{C})} - \frac{(u_5-\ell_5)}{(\mathfrak{C})} \right\|$$

And

$$\| \ell_0 - \ell_5 \|^2 \| \leq \mathcal{R}^2 \left(2 - \frac{2}{\mathfrak{C}_0\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3\mathfrak{C}_4\mathfrak{C}_5} (u_0 - \ell_0, u_1 - \ell_1, u_2 - \ell_2, u_3 - \ell_3, u_4 - \ell_4, u_5 - \ell_5) \right)$$

$$= \mathcal{R}^2 \left(2 + \frac{\|\ell_0-\ell_5\|^2 + \|u_0-u_5\|^2 - \|u_5-\ell_0\|^2 - \|u_0-\ell_5\|^2 +}{\mathfrak{C}_0\mathfrak{C}_5} \right)$$

(2.8)

Let $\alpha = \| \ell_0 - \ell_5 \|, and\ \varepsilon = \| u_0 - u_5 \|$. By Proposition 1.1 we obtain that

$$\ell_5 \in \mathcal{A} \subset B_{\mathcal{R}}\left(\ell_0 - \mathcal{R}\frac{u_0-\ell_0}{\mathfrak{C}_0}\right).$$

Put $z = \ell_0 - \mathcal{R}\frac{c}{\mathfrak{C}_0}$.Note that $\angle u_0\ell_0\ell_5 = \pi - \angle z\ell_0\ell_5, \| z - \ell_5 \| \leq \mathcal{R}$. By by the theorem of cosine we take

$$\begin{aligned} \cos\angle u_0\ell_0\ell_5 &= -\cos\angle z\ell_0\ell_5 \\ &= \frac{-\mathcal{R}^2 + \alpha^2 - \| z - \ell_5 \|}{2\mathcal{R}\alpha} \\ &\leq -\frac{\alpha}{2\mathcal{R}} \end{aligned}$$

And

$$\| u_0 - \ell_5 \|^2 = \| u_0 - \ell_0 \|^2 \; 2 \; + \| \ell_0 - \ell_5 \|^2$$
$$- \; 2 \, \| u_0 \; - \; \ell_0 \| \; \cdot \; \| \ell_0 \; - \ell_5$$
$$\| \; cos\angle u_0 \ell_0 \ell_5$$
$$\geq \; \mathfrak{C}_0^{\;2} \; + \alpha^2 \; + \frac{\alpha^2 \mathfrak{C}_0}{\mathcal{R}}$$

Similarly, we can derive that,

$$\| u_0 - \ell_5 \|^2 \geq \; \mathfrak{C}_0^{\;2} \; + \alpha^2 \; + \frac{\alpha^2 \mathfrak{C}_0}{\mathcal{R}}.$$

Using Formula (2.8), we obtain,

$$\alpha^2 \leq \; \mathcal{R}^2 \left(2 + \frac{\alpha^2 + \varepsilon^2 - \; \mathfrak{C}_0^{\;2} - \alpha^2 - \frac{\alpha^2 \mathfrak{C}_0}{\mathcal{R}} - \; \mathfrak{C}_0^{\;2} - \alpha^2 - \frac{\alpha^2 \mathfrak{C}_0}{\mathcal{R}}}{\mathfrak{C}_0 \mathfrak{C}_5} \right)$$

and following the transformations

$$\alpha \leq \frac{\mathcal{R}}{\sqrt{(\mathcal{R} + \mathfrak{C}0)(\mathcal{R} + \mathfrak{C}1)(\mathcal{R} + \mathfrak{C}2)(\mathcal{R} + \mathfrak{C}3)(\mathcal{R} + \mathfrak{C}4)(\mathcal{R} + \mathfrak{C}5)}} \cdot \sqrt{\| u_0 - u_5 \|^2 - \mathfrak{C}0 - \mathfrak{C}1 - \mathfrak{C}2 - \mathfrak{C}3 - \mathfrak{C}4 - \mathfrak{C}5)^2)}$$

Observation 2.1: If the value of u_0 is contained within the set $\mathcal{A}$, resulting in $\mathfrak{C}_0 = 0$, then Formula (2.7) can be considered reliable. As a result, $\ell_0 \; = \; \mathcal{P}_{\mathcal{A}} u_0 = u_0$ and by applying Proposition 1.1, we obtain

$$u_0 \; \in \; \mathcal{A} \; \subset \; B_{\mathcal{R}} \left(\ell_5 - \mathcal{R} \frac{u_5 - \ell_5}{\mathfrak{C}_5} \right).$$

Hence, $cos\angle u_0 \ell_0 \ell_5 \leq - \frac{\| u_0 - \ell_5 \|}{2\mathcal{R}}$, as demonstrated in the execution of Theorem 2.2. Applying the law of cosines to the triangle $u_0 \ell_0 \ell_5$, we deduce that,

$$\| u_0 - u_5 \|^2 = \| u_0 - \ell_5 \|^2 + \; \mathfrak{C}_5^{\;2} - \; 2$$
$$\| u_0 \; - \; \ell_5 \| \; \mathfrak{C}_1 cos\angle u_0 \ell_0 \ell_1$$
$$\geq \| u_0 - \ell_5 \|^2 + \; \mathfrak{C}_5^{\;2}$$
$$+ \frac{\mathfrak{C}_5 \, \| u_0 - \ell_5 \|^2}{\mathcal{R}}$$

The final equation is equivalent to Formula (2.7) with $\mathfrak{C}_0 = 0$ and $u_0 = \ell_0$.

The observation can be made estimation that the (2.7) for any convex strongly set with radius R aligns estimation with the (1.2) for a ball of the same radius.

Corollary 2.1. The closed convex subset A ⊂ H is a closed subset. It is possible to compare the following characteristics:

Strongly convex with a radius greater than zero is the definition of the set denoted by the letter A.

(1)

since $\mathfrak{C} > 0$ and this points $u_0, u_5 \in \mathrm{H} \backslash \mathrm{U}(\mathcal{A}, \mathfrak{C})$, the following inequality is satisfied, where

$\ell_i = \mathcal{P}_{\mathcal{A}} u_i$ for i = 0, 1.

(2)

$$\| \ell_0 - \ell_5 \| \leq \frac{\mathcal{R}}{\mathcal{R} + \mathfrak{C}} \| u_0 - u_5 \|.$$

Consider a bounded subset $\mathcal{A} \subset \mathfrak{B}$, and define $k_{\mathcal{A}} = [diam\, \mathcal{A}] + 1$,where $[t]$ represents the largest integer less than or equal to t. A Banach space B is said to satisfy the intersection property of Mazur if every closed, convex, and bounded subset of B can be represented as the intersection of all closed balls that enclose it. Within the context of such spaces, this characteristic offers a key characterization that governs how closed balls cross with one another. It was proved by Mazur that this condition is valid for every reflexive Banach space that is accompanied by a Fréchet differentiable norm on its unit sphere. As a result of his work, it was proven that the existence of this differentiable norm guarantees the satisfaction of the Mazur intersection property. This provided significant insights into the geometric structure of reflexive Banach spaces.

Let us use the notation $strco_{\mathcal{R}}\, \mathcal{A}$ to identify a subset A⊂B with a diameter that is smaller than or equal to R. This will allow us to represent the highly convex hull of radius R [2,18,4].

$$strco_{\mathcal{R}}\, \mathcal{A} = \bigcap_{u \in \mathfrak{B}:\, \mathcal{A} \subset B_{\mathcal{R}}(u)} B_{\mathcal{R}}(u).$$

According to [11], if $diam\, \mathcal{A} \leq \mathcal{R}_1 < \mathcal{R}_2$, then:"

$$\mathcal{A} \subset strco_{\mathcal{R}_2}\, \mathcal{A} \subset strco_{\mathcal{R}_1}\, \mathcal{A}. \quad (2.9)$$

"Using Formula (2.10) and the definition of the strongly convex hull, the Mazur intersection property for any arbitrary closed, convex, and bounded subset $\mathcal{A} \subset \mathfrak{B}$ can be expressed as:"

$$\bigcap_{k = k_{\mathcal{A}}}^{\bowtie} strco_k\, \mathcal{A} = \mathcal{A}$$

Let $\mathcal{A}$ represent the weak closure of the subset A⊂B.

It is possible to satisfy the requirement of weak compactness in any reflexive Banach space by ensuring that every convex, closed, and subset which is bounded is present. This outcome is a well-known and direct consequence of the Banach-Alaoglu theorem, which guarantees that the closed ball which is unit in the space of dual is weak compact. This result is a direct consequence of the concept. The weak compactness of subsets in reflexive Banach spaces becomes an essential and fundamental quality within the realm of functional analysis as a corollary to this idea.

Lemma 2.2: Accept that B is a Banach space that is reflexive & possesses the intersection property of Mazur.

Let the subset A⊂B is a convex, closed, and function which is bounded. Then, for any unit vector P that belongs to B*, we have

$$\lim_{k \to \infty} s(\mathcal{P}, strco_k\, \mathcal{A}) = s(\mathcal{P}, \mathcal{A}).$$

Proof: Consider the possibility that the statement is not true. Without sacrificing generality, we can make the assumption that there is a set of points P_0 that belong to the set B_1*(0) and ε_0 that is greater than zero.

Consider the possibility that the statement is not true. Without sacrificing generality, we can make the assumption that there is a set of points $\mathcal{P}_0$ that belong to the set $\partial B_1^*(0)$ and ε_0 that is greater than zero.

$$s(\mathcal{P}_0, strco_k\, \mathcal{A}) - s(\mathcal{P}_0, \mathcal{A}) \geq \varepsilon_0, for\ all\ k \geq kA. \quad (2.10)$$

The space of reflexivity B implies the set $strco_k\, \mathcal{A}$ is weakly compact for any k ≥ kA. Examine a point ℓ_k in $strco_k\, \mathcal{A}$ where $(\mathcal{P}_0, \ell_k)$= $s(\mathcal{P}_0, strco_k\, \mathcal{A})$. In addition, consider a $\ell_k \in \mathcal{A}$ such that $(\mathcal{P}_0, \ell_0)$= $s(\mathcal{P}_0, \mathcal{A})$.

The Eberlein-Mulian theorem suggests that the bounded sequence $\{\ell_k\}_{k=1}^{\infty}$ weakly converges to a point $b_0 \in \mathfrak{B}$ without losing generality.

Using inequality (2.11), it follows that we have:

$$(\mathcal{P}_0, b_0) = \lim_{k \to \infty} (\mathcal{P}_0, \ell_k) \geq (\mathcal{P}_0, \ell_0) + \varepsilon_0 = s(\mathcal{P}_0, \mathcal{A}) + \varepsilon_0.$$

i.e. $b0 \notin \mathcal{A}$.

In contrast to what was stated in the past, the intersection property of Mazur and (2.10) are the ones who are accountable for reaching that result.

$$b0 \in \bigcap_{k = k_{\mathcal{A}}}^{\infty} \mathrm{Cl}_w\{\ell_k\}_{k=m}^{\infty} \subset \bigcap_{m = k_{\mathcal{A}}}^{\infty} strco_m\, \mathcal{A} = \mathcal{A}.$$

If the unit sphere in the Banach space does not contain any line segments that are not degenerate, then it can claim that the space is strictly convex.

Let $\omega: [0, +\infty) \rightarrow [0, +\infty)$ be a nonnegative function such that $\omega(0) = \lim_{t \to +0} \omega(t) = 0.$

It is a statement that the Banach space of metric projection in a $\mathfrak{B}$ is uniformly continuous with modulus ω on a class of sets $A \subset 2^{\mathfrak{B}}$ if, for each set $\mathcal{A} \in \mathbb{A}$ and any points u_0, $u_1 \in \mathfrak{B} \setminus \mathcal{A}$, the following holds true:

$$\| \ell_0 - \ell_5 \| \leq \omega(\| u_0 - u_5 \|),$$

Where, $\ell_i = \mathcal{P}_{\mathcal{A}} u_i, i = 0,1,2,3,4,5$

"In the Hilbert space, the metric projection is uniformly continuous on the class of closed convex sets with the modulus $\omega(t) = t$."

III. APPLICATIONS

Take into consideration the following example of minimisation in the following order:

$$\text{in } f(u) \quad u \in \mathcal{A} \subset \mathcal{R}_n. \tag{3.11}$$

We will discuss the gradient projection algorithm in its standard form: $u_1 \in \partial \mathcal{A}$,

$u_{k+1} = \mathcal{P}_{\mathcal{A}}(u_k - \alpha_k \alpha k f'(u_k)), \alpha_k > 0.$

Consider the following assumptions:

1. Let $U \subset \mathbb{R}^n$ be subset arbitrary such that $\mathcal{A} = \cap_{u \in U,} B_{\mathcal{R}}(u) = \emptyset$represents a closed ball of radius R centred at u.
2. The function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and convex with its gradient $f'(u)$ satisfying the Lipschitz condition. Specifically, there exists a constant L>0 such that for all $u, y \in \mathbb{R}^n$,

   $$\| f'(u) - f'(y) \| \leq L \| u - y \|.$$

   This condition ensures that the gradient does not change too rapidly and is Lipschitz continuous with the constant LL.
3. A natural number for any k, £ a unit vector P $(u_k) \in N(A,u_k)$ in $N(A,u_k)$ such that the inner product $(P(u_k), f'(u_k)) < 0$. This implies that $u_k - \alpha_k f'(u_k) \notin A$ for any $\alpha_k > 0$), meaning that stepping in the negative gradient direction scaled by α_k takes the point outside of the set $\mathcal{A}$.
4. The solution to problem (3.11) is denoted u*, which lies on the boundary of the set A, i.e., $u_* \in \partial \mathcal{A}$. This condition highlights the geometric relationship between the solution and the set AA.

It is worth noting that state (ii) is corresponding to the statement that for all $u, y \in \mathbb{R}^n$, the gradient $f'(u)$obeys the Lipschitz property, as outlined in [9].

$$(u - y, f'(u) - f'(y)) \geq \frac{1}{L} \| f'(x) - f'(y) \|^2; \tag{3.12}$$

Concepts on Lipschitz continuity have important implications for algorithm design and optimization in many domains. The results help gradient-based optimization methods like gradient descent and Nesterov's accelerated methods achieve quicker convergence rates and better stability by tightening Lipschitz constraints with strong convexity. Lipschitz continuity improves machine learning model stability and robustness. Lipschitz regularization controls learning smoothness and makes neural networks more resistant to adversarial attacks and better at generalizing to unknown data. Lipschitz constraints help control systems allocate resources and establish robust feedback mechanisms for dynamic stability. Monte Carlo simulations and finite-difference approximations can estimate Lipschitz constants for complicated, high-dimensional systems. These advances apply theoretical ideas to structural stability, data processing, and optimization challenges.

IV. CONCLUSION

This paper uses robust projection methods and metric analysis to provide a comprehensive evaluation of the impact of an additional constant on the Lipschitz condition. By examining the characteristics of exceedingly convex sets in Banach spaces, the article demonstrates that better bounds for the Lipschitz constant have been discovered. It continues by demonstrating the implications of these constraints on the consistency and linearity of metric forecasts. Convex optimization is a method for characterizing and assuring the stability of solutions in various mathematical and computing contexts; these results shed fresh light on this topic. This theoretical understanding of Lipschitz continuity is enhanced to optimization and control problems which are set up on solid foundations by these results. They are all benefits that have resulted from the findings. Researching the algorithmic implementation of these findings in real-world systems, as well as other potential real-world applications, holds great promise. There are several potential uses for these results.

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